Exercise 1 (15 pts) Use Divergence Theorem to find the outward flux

$$\int\int_S \vec{F} \cdot \vec{n}\, d\sigma$$

of the vector field

$$\vec{F} = 5x\,\vec{i} + 2y\,\vec{j} + 8z\,\vec{k}$$

across the sphere $\rho = 2$

Solution : $\vec{F} = 5x\vec{i} + 2y\vec{j} + 8z\vec{K} \Rightarrow$ so $\vec{F}$ is continuous and no bad points. and $\mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(8z) = 5+2+8$
$= 15$

By Divergence Theorem, $\iint_S \vec{F}.\vec{n}\, d\sigma = \iiint_V \mathrm{div}(\vec{F})\, dV$

$$= \iiint_V 15\, dV$$

$$= 15\,(\text{Volume})$$

$$= 15\left(\frac{4}{3}\pi(2)^3\right)$$

$$= 160\pi.$$

Exercise 2 (15 pts) Let $\vec{F} = 5y\,\vec{i} + 8x\,\vec{j} + z\,\vec{k}$ and S be the paraboloid

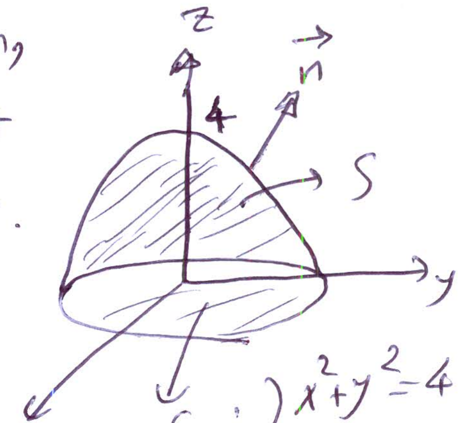
$$z = 4 - x^2 - y^2,\ z \geq 0 \longrightarrow div(\vec{F}) = 0 + 0 + 1 = 1$$

which is open at the bottom. Find

$$\int\int_S \vec{F} \cdot \vec{n}\, d\sigma$$

(where $\vec{n}$ is the outer normal vector to our surface) by using Divergence Theorem

Solution: Since S is open at the bottom, we add S_1 $(z=0,\ x^2+y^2=4)$ to make it closed and to get ballance, substract it.

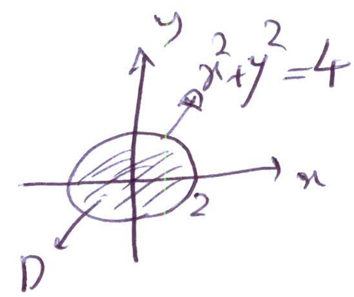


$S_1 : \begin{cases} x^2+y^2=4 \\ z=0 \end{cases}$

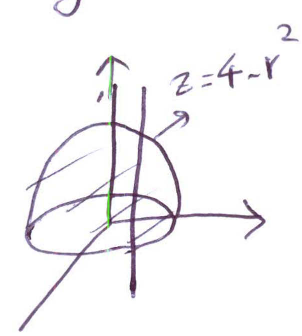
the normal vector for S_1 $= \vec{n} = -k$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \left[\left(\iint_S \vec{F}\cdot\vec{n}\,d\sigma\right) + \left(\iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma\right)\right] - \left(\iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma\right)$$

$$= \underbrace{\iint_{S\cup S_1}}_{\text{closed, so we can use Divergence Thrm}} \vec{F}\cdot\vec{n}\,d\sigma - \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma$$



$$= \iiint_V div(\vec{F})\,dv - \iint_D (5y\vec{i}+8x\vec{j}+z\vec{k})\cdot(-\vec{k})\,dxdy$$



$$= \iiint_V dv - \cancel{\iint_D -z\,dxdy}\ 0 \qquad \begin{cases} z=0 \\ x^2+y^2=4 \end{cases}$$

$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

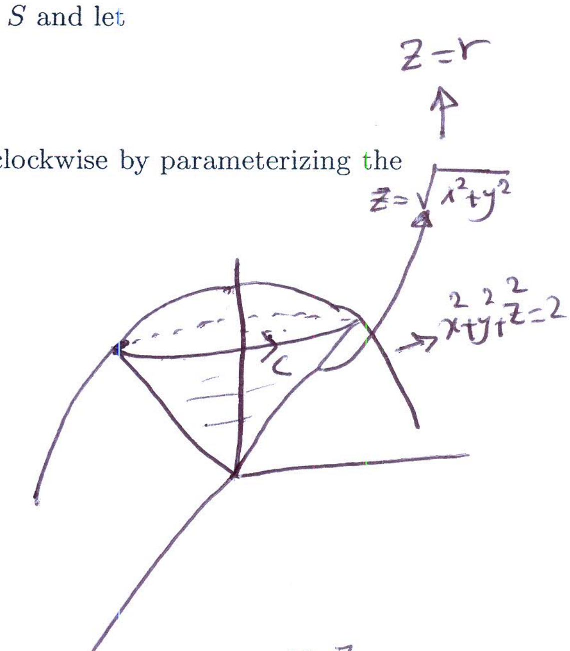
$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta = \cdots$$

Exercise 3 Let S be the upper spherical cap formed by cutting the sphere $x^2+y^2+z^2=2$ with a cone having the equation $z=\sqrt{x^2+y^2}$. Answer the following questions:

a) (10 pts) Let C denote the boundary of the surface (the cap) S and let

$$\vec{F}=(z-y)\,\vec{i}+y\,\vec{k}$$

Calculate the circulation of $\vec{F}$ around the curve C counter clockwise by parameterizing the curve C

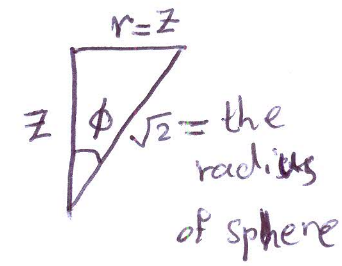


Solution :

$C=$ the circle $(z=1,\ x^2+y^2=1)$

$=$ the circle with radius 1 and center $(0,0,1)$

$$\Rightarrow C:\begin{cases} x=\cos\theta \Rightarrow dx=-\sin\theta\, d\theta \\ y=\sin\theta\,,\quad 0\leq\theta\leq 2\pi \\ z=1 \Rightarrow dz=0 \end{cases}$$



$$\tan\phi=\frac{r}{z}=\frac{z}{z}=1$$

$$\Rightarrow \phi=\frac{\pi}{4}$$

$$z^2+r^2=(\sqrt{2})^2=2$$

$$z^2+z^2=2$$

$$\Rightarrow \boxed{z=r=1}$$

$$\oint_C \vec{F}\cdot\vec{dr}=\oint_C M\,dx+N\,dy+P\,dz$$

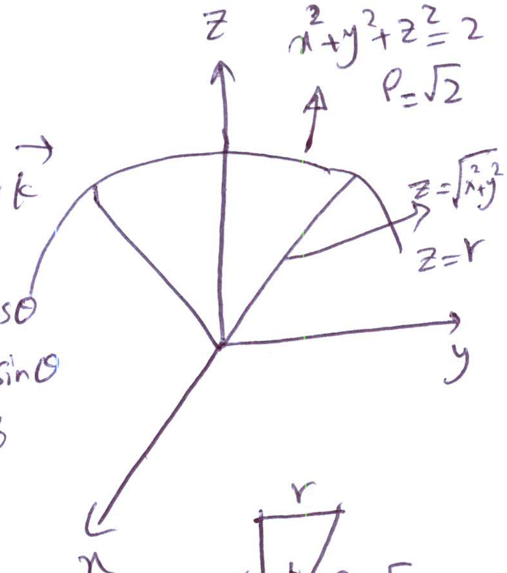
$$=\int_0^{2\pi}(z-y)\,dx+0\,dy+\cancel{y\,dz}_{0}$$

$$=\int_0^{2\pi}(1-\sin\theta)(-\sin\theta)\,d\theta=\int_0^{2\pi}(-\sin\theta+\sin^2\theta)\,d\theta=\int_0^{2\pi}\left(-\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\left[\cos\theta+\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}=\left[\cos 2\pi+\frac{1}{2}(2\pi)-\frac{1}{4}\cancel{\sin(2(2\pi))}_{0}\right]-\left[\cos 0+\frac{1}{2}(0)-\frac{1}{4}\cancel{\sin 0}_{0}\right]$$

$$=1+\pi-1=\boxed{\pi}$$

b) (10 pts) Solve part a) by using Stokes' Theorem

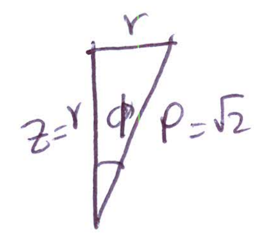


$$\overrightarrow{S(\phi,\theta)} = \sqrt{2}\sin\phi\cos\theta\,\vec{i} + \sqrt{2}\sin\phi\sin\theta\,\vec{j} + \sqrt{2}\cos\phi\,\vec{k}$$

$$0 \le \phi \le \pi/4, \quad 0 \le \theta \le 2\pi$$

by using the spherical coordinates: $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

$$\vec{S}_\phi \times \vec{S}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix} = \vec{i}(2\sin^2\phi\cos\theta) + \vec{j}(2\sin^2\phi\sin\theta) + \vec{k}(\sin 2\phi)$$



$$\tan\phi = \frac{r}{z} = \frac{r}{r} = 1$$

$$\Rightarrow \boxed{\phi = \pi/4}$$

$$\operatorname{Curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & 0 & y \end{vmatrix} = \vec{i}(1-0) - \vec{j}(0-1) + \vec{k}(0-(-1)) = \vec{i} + \vec{j} + \vec{k}$$

$$\iint_S \operatorname{curl}(\vec{F})\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/4} (\vec{i}+\hat{j}+\vec{k})\cdot\left(2\sin^2\phi\cos\theta\,\vec{i} + 2\sin^2\phi\sin\theta\,\vec{j} + \sin 2\phi\,\vec{k}\right) d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left(2\sin^2\phi(\cos\theta+\sin\theta) + \sin 2\phi\right) d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[(\cos\theta+\sin\theta)\left(\phi - \frac{1}{2}\sin 2\phi\right)_0^{\pi/4} + \left(-\frac{1}{2}\cos 2\phi\right)_0^{\pi/4}\right] d\theta$$

$$= \int_0^{2\pi} \left[(\cos\theta+\sin\theta)\left(\frac{\pi}{4}-\frac{1}{2}\right) + \frac{1}{2}\right] d\theta = \left(\frac{\pi}{4}-\frac{1}{2}\right)(\sin\theta - \cos\theta)_0^{2\pi} + \left(\frac{1}{2}\theta\right)_0^{2\pi}$$

$$= \cancel{\left(\frac{\pi}{4}-\frac{1}{2}\right)\left[(0-1)-(0-1)\right]} + \pi = \boxed{\pi}$$

Exercise 4 (10 pts) Solve the IVP:

$$x\frac{dy}{dx} - 2y = x^3 + 3, \ y(1) = 2$$

$\frac{dy}{dx} - \frac{2}{x}y = x^2 + \frac{3}{x}$, $y(1) = 2$

$I(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

$x^{-2}\frac{dy}{dx} - 2x^{-3}y = 1 + 3x^{-3}$

$(x^{-2}y)' = 1 + 3x^{-3} \Rightarrow x^{-2}y = 1 - \frac{3}{2}x^{-2} + C$

$\xRightarrow[y=2]{x=1} 1^{-2}(2) = 1 - \frac{3}{2}(1)^{-2} + C$

$\Rightarrow 2 = 1 - \frac{3}{2} + C$

$\Rightarrow \frac{5}{2} = C$

$\Rightarrow \boxed{x^{-2}y = 1 - \frac{3}{2}x^{-2} + \frac{5}{2}}$